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Minimal electromagnetic coupling in elementary quantum mechanics; a group theoretical derivation

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Abstract. The equation of motion of a spinless quantum particle in an external uniform parallel electromagnetic field is derived from the symmetry of the field by group theoretical techniques. The minimal coupling is obtained directly, without a detour via the free particle equation. This derivation, which is valid in a Galilean as well as in a Poincaré framework, is a contribution to the recasting process of quantum mechanics in the spirit of a paper by Lévy-Leblond.

1. Introduction

In the recasting process of quantum mechanics the traditional, but heuristic, correspondence principle is gradually being replaced by the invariance principle. This is mainly due to the work of Lévy-Leblond (1971, 1974, 1975) who derived, in particular, the Schrödinger equation of a free particle from Galilean invariance just like the free Klein–Gordon equation can be derived from Poincaré invariance, as has been (implicitly) known since Wigner (1939). The next step in that recasting process obviously leads to the principle of minimal electromagnetic coupling. This principle says that the equation of motion of a charged particle in an external electromagnetic field will be obtained from the free equation by the substitutions

$$\mathrm{i}\partial_t \to \mathrm{i}\partial_t - e\Phi \qquad \text{and} \qquad -\mathrm{i}\nabla \to -\mathrm{i}\nabla - e\boldsymbol{A}$$

where Φ and $\boldsymbol{A}$ are the scalar and vector potentials of the field.

Traditionally the minimal coupling has been introduced in quantum mechanics by an argument of correspondence to classical mechanics. This is still the usual textbook 'derivation'. In the course of the recasting process another derivation has been given by Jauch (1964) who introduced an argument of pseudo-invariance under 'instantaneous' Galilei transformations. His derivation has been remodelled, generalised and criticised by several authors (Jauch 1968, Lévy-Leblond 1970, 1971, 1974, Piron 1972, Celeghini *et al* 1976, Kraus 1977) but a satisfactory analogue for Poincaré relativistic quantum systems has not been given.

Jauch's derivation has in common with the traditional one, that it starts from the free particle equation. The arguments by which the free equation is modified are different: (pseudo-)invariance principle versus correspondence principle. Although Jauch's pseudo-invariance principle may be the best argument available in the general

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case of *arbitrary* fields breaking all space–time symmetry, it is in fact a 'waste' of invariance in important cases of *special* fields preserving a part of that symmetry.

Particularly, a uniform (i.e. homogeneous in space and constant in time) electromagnetic field is invariant under a six-parameter subgroup of the ten-parameter Galilei or Poincaré symmetry group of the empty space–time (Janner and Ascher 1970, Bacry *et al* 1970a, b, see also the extensive discussion on Galilean electromagnetism in Le Bellac and Lévy-Leblond 1973). In this paper the equation of motion of a spinless charged particle in a uniform external field will be derived from that six-parameter subgroup just as the free equation can be derived from the full ten-parameter symmetry group, in the spirit of Lévy-Leblond's (1974) approach. The minimal coupling will be obtained directly, without a detour via the free particle equation. The point is that this result follows from *exact* rather than from *pseudo*-invariance and that it is valid in a Galilean as well as in a Poincaré framework.

The restriction to *uniform* fields is the price that has to be paid for keeping 'enough' space–time symmetry so that group theory can be usefully applied. As the invariance group of a uniform field does not contain the three-dimensional rotation group one cannot expect to find the spin as a quantum number; this explains the restriction to spinless particles.

Actually, the derivation will be given here only for uniform parallel fields ($\boldsymbol{E}\|\boldsymbol{B}$; possibly with $\boldsymbol{E}$ or $\boldsymbol{B}$ vanishing). This restriction is less stringent than it seems to be, because most uniform fields can be brought into this form by a suitable coordinate transformation, the only exceptions being the uniform crossed fields ($\boldsymbol{E}\perp\boldsymbol{B}$ and $\boldsymbol{E}^2=\boldsymbol{B}^2\neq 0$) in a Poincaré frame.

The group G that we will consider in this paper leaves a uniform electromagnetic field with $\boldsymbol{E}$ and $\boldsymbol{B}$ parallel to the z axis invariant. However, G is not the full symmetry group of a parallel field, as we will disregard the inversions (and in the Galilean case of a pure electric field we will even disregard the pure Galilei transformations in the x–y plane). The representations of G were classified by Bacry *et al* (1970a, b) who also noted the relation of these representations to the equation of motion.

This paper is organised as follows. In § 2 all those ingredients are assembled that can be obtained by group theoretical methods only. In § 3 a description in configuration space is given and the equations of motion are derived. The last section contains a discussion on the physical input in § 3.

The present derivation results from a more comprehensive investigation of representations of symmetry groups in quantum mechanics (Hoogland 1977).

2. Group theoretical preliminaries

Let G be the subgroup of the Galilei (or Poincaré) group generated by the translations in space–time, the rotations around the z axis and the pure Galilei (or Lorentz) transformations in the z direction. This group G will be called briefly the *invariance group*, as it leaves a uniform electromagnetic field with $\boldsymbol{E}$ and $\boldsymbol{B}$ parallel to the z axis invariant.

In the present case it is quite handy to use a Minkowski-like two-vector notation in the t–z plane and the usual vector notation in the x–y plane. By this notation the Galilei and Poincaré groups can be treated simultaneously and the structure of G is manifestly displayed as the direct product of the one-(space-) dimensional Galilei (or

Poincaré) group (notation $\mathscr{G}^{(1)}$ or $\mathscr{P}^{(1)}$) operating in the t–z plane and the two-dimensional Euclidean group (notation $\mathscr{E}^{(2)}$) operating in the x–y plane. So the space–time events x and the group elements g are denoted by

$$x = (x^\mu; \boldsymbol{x}), \qquad g = (a^\mu, \chi; \boldsymbol{a}, \phi) \tag{1}$$

where

$$x^\mu = \begin{pmatrix} t \\ z \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad a^\mu = \begin{pmatrix} a_t \\ a_z \end{pmatrix}, \quad \boldsymbol{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}. \tag{2}$$

Here the a are the translations; ϕ is the rotation angle;

$$\chi = \begin{cases} v & \text{(Gal)} \\ \tanh^{-1} v & \text{(Poin)} \end{cases} \tag{3}$$

where v is the velocity of the pure Galilei (or Lorentz) transformations. The group G operates on space–time by

$$gx = (\Lambda^\mu{}_\nu(\chi)x^\nu + a^\mu; R(\phi)\boldsymbol{x} + \boldsymbol{a}) \tag{4}$$

where

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \tag{5}$$

and

$$\Lambda^\mu{}_\nu(\chi) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \chi & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} & \text{(Gal)} \\ \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix} = (1-v^2)^{-1/2}\begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} & \text{(Poin).} \end{cases} \tag{6}$$

The group product reads

$$g'g = (a'^\mu + \Lambda^\mu{}_\nu(\chi')a^\nu, \chi' + \chi; \boldsymbol{a}' + R(\phi')\boldsymbol{a}, \phi' + \phi). \tag{7}$$

The group elements can be expressed in terms of the six infinitesimal generators P_t, P_x, P_y, P_z, J and K by

$$g = \exp(\mathrm{i}a^\mu P_\mu)\exp(\mathrm{i}\chi K)\exp(-\mathrm{i}\boldsymbol{a}\cdot\boldsymbol{P})\exp(-\mathrm{i}\phi J) \tag{8}$$

where $a^\mu P_\mu = a_t P_t - a_z P_z$ and $\boldsymbol{a}\cdot\boldsymbol{P} = a_x P_x + a_y P_y$. The non-vanishing commutators are

$$[P_t, K] = -\mathrm{i}P_z \tag{9}$$

$$[P_z, K] = -\mathrm{i}P_t \qquad \text{(Poin)} \tag{10a}$$

$$[J, P_x] = \mathrm{i}P_y \tag{11}$$

$$[J, P_y] = -\mathrm{i}P_x. \tag{12}$$

In order to obtain the *projective* representations of a group one has to go through a rather technical procedure of extension of its Lie algebra and calculation of its exponents (Bargmann 1954, Lévy-Leblond 1971, 1974). Fortunately, in the present case one can profit from the structure of the six-parameter group G as a direct product of the two three-parameter subgroups $\mathscr{G}^{(1)}$ (or $\mathscr{P}^{(1)}$) and $\mathscr{E}^{(2)}$. The Lie algebras of these subgroups can be extended very easily indeed (Lévy-Leblond 1969, 1974). After a redefinition of the translation generators the result is given by (9), (13) and (10*b*) (or (10*a*)) for $\mathscr{G}^{(1)}$ (or $\mathscr{P}^{(1)}$) and by (11), (12) and (14) for $\mathscr{E}^{(2)}$ (ϵ, β and μ are reals):

$$[P_t, P_z] = -\mathrm{i}\epsilon \qquad (13)$$

$$[P_z, K] = -\mathrm{i}\mu \qquad \text{(Gal)} \qquad (10b)$$

$$[P_x, P_y] = -\mathrm{i}\beta. \qquad (14)$$

By splitting the Lie algebra of G in two 'disjoint' subalgebras before the extension process, we have omitted the 'mixed' commutators $[K, J] = \lambda$ and (only for Galilei) $[P_t, J] = \lambda'$, although these do not vanish in a general extension of the whole Lie algebra of G. However, not all extensions of the Lie algebra of G correspond to extensions of the multiply connected (!) Lie group G itself. (This is analogous to the situation for the two-dimensional Galilei group: see p 240 of Lévy-Leblond (1971) and § 5 of de Swart (1974).) Only those extensions in which the mixed commutators vanish play a role in the determination of the projective representations of the invariance group G, as can be shown by the following argument. The element g_1 of the form $(0^\mu, 0; \mathbf{0}, 2\pi)$ corresponds to the unit element in the invariance group G; it belongs to the kernel of the covering epimorphism from G^* (the covering group of G) onto G. Hence, from § 3e of Bargmann (1954) it follows that the operator $U(g_1) = \exp(-\mathrm{i}2\pi J)$ has to be a multiple of the unit operator. In particular, $\exp(-\mathrm{i}2\pi J)$ should commute with the generators K and P_t. On the other hand, from $[K, J] = \lambda$ it follows that $\exp(-\mathrm{i}2\pi J)K \exp(+\mathrm{i}2\pi J) = K + \mathrm{i}2\pi\lambda$. This proves that $\lambda = 0$. Hence, the mixed commutators $[K, J]$ and, analogously, $[P_t, J]$ vanish. The extended Lie algebra of the invariance group G is thus given by (9)–(14) (see also Combe and Richard 1973).

It can be checked easily that the operators $C_\parallel$ and $C_\perp$ defined by

$$C_\parallel = \begin{cases} 2\mu P_t - P_z^2 + 2\epsilon K & \text{(Gal)} \\ P_t^2 - P_z^2 + 2\epsilon K & \text{(Poin)} \end{cases} \qquad (15)$$

$$C_\perp = P_x^2 + P_y^2 - 2\beta J \qquad (16)$$

are invariants (Casimir operators) of the extended Lie algebra (9)–(14), i.e. they commute with all generators. Hence, in an irreducible (unitary) representation these are (real) multiples of the unit operator. Combination of (15) and (16) gives

$$\begin{matrix} \text{(Gal)} & 2\mu P_t \\ \text{(Poin)} & P_t^2 \end{matrix} \Bigg\} = P_x^2 + P_y^2 + P_z^2 - 2(\epsilon K + \beta J) + C_\parallel - C_\perp. \qquad (17)$$

This relation will result in the equation of motion, as soon as the explicit expressions for the generators in configuration space are known (see the next section).

The group exponents ξ occurring in a projective representation U by

$$U(g')U(g) = (\exp \mathrm{i}\xi(g', g))U(g'g) \qquad (18)$$

can be calculated from the extended Lie algebra.

In a sophisticated method for the calculation of group exponents one uses the Campbell–Baker–Hausdorff relation (Lévy-Leblond 1971, p 241). A more 'down to earth' method by explicit matrix expressions has been given by Lévy-Leblond (1974, p 111). Both methods can be applied straightforwardly in the present case. Due to the above-mentioned 'disjointness' property of the extended subalgebras the exponents ξ of G are a linear combination of the exponents of $\mathscr{G}^{(1)}$ (or $\mathscr{P}^{(1)}$) and $\mathscr{E}^{(2)}$, so the actual calculation of the exponents may be carried out separately for these subgroups. The explicit expressions that will be obtained for the exponents depend on the phase convention adopted for the operators $U(g)$. For that reason different group parametrisations give different, but equivalent (Bargmann 1954), exponents.

With the parametrisation (8) and the extended Lie algebra (9)–(14) one obtains (see also Lévy-Leblond 1969, Combe and Richard 1973)

$$\xi = \epsilon\xi_1 + \beta\xi_2 + \begin{cases} \mu\xi_0 & \text{(Gal)} \\ 0 & \text{(Poin)} \end{cases} \tag{19}$$

with

$$\xi_0(g', g) = \tfrac{1}{2}v'^2 a_t + v' a_z \tag{20}$$

$$\xi_1(g', g) = \begin{cases} \tfrac{1}{2}(a'_z a_t - a'_t a_z - a'_t v' a_t) & \text{(Gal)} \\ \tfrac{1}{2}(a'_z a_t - a'_t a_z)\cosh\chi' + \tfrac{1}{2}(a'_z a_z - a'_t a_t)\sinh\chi' & \text{(Poin)} \end{cases} \tag{21}$$

$$= \tfrac{1}{2}\epsilon_{\mu\sigma} a'^{\sigma} \Lambda^{\mu}{}_{\tau}(\chi') a^{\tau} \qquad \text{with } \epsilon_{\mu\sigma} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\xi_2(g', g) = \tfrac{1}{2}(\boldsymbol{a}' \times R(\phi')\boldsymbol{a})_z. \tag{22}$$

3. Description in configuration space

Consider a projective representation $U(g)$ of the invariance group, operating on wavefunctions in configuration space as follows:

$$(U(g)\psi)(gx) = (\exp \mathrm{i}\theta(g; x))\psi(x). \tag{23}$$

The physical meaning of this equation will be discussed in the conclusion. The phase function θ in (23) is related to the exponent ξ in (18). An easy calculation gives

$$\xi(g', g) = \theta(g'; gx) + \theta(g; x) - \theta(g'g; x). \tag{24}$$

Without loss of generality one may assume that $\theta(g; x=0)$ vanishes. (If it does not vanish the operators $U(g)$ may be redefined by multiplication by the phase factor $\exp(-\mathrm{i}\theta(g; 0))$, allowed by the projectivity of the representation.) Substitution of $x = 0$ in (24) gives then

$$\xi(g', g) = \theta(g'; g0). \tag{25}$$

Let h_x be the translation from the origin ($x = 0$) to the space–time event x. Substitution of (g, h_x) for (g', g) in (25) gives

$$\xi(g, h_x) = \theta(g; x). \tag{26}$$

Hence, without loss of generality one may assume that a projective representation in configuration space has the form

$$(U(g)\psi)(gx) = (\exp \mathrm{i}\xi(g, h_x))\psi(x). \tag{27}$$

The translation h_x is the group element (see equation (1)) with $\chi = 0$, $\phi = 0$, $a^{\mu} = x^{\mu}$ and $\boldsymbol{a} = \boldsymbol{x}$. Substitution of this in the equations (19)–(22) gives

$$\xi(g, h_x) = \tfrac{1}{2}\beta(\boldsymbol{a} \times R(\phi)\boldsymbol{x})_z + \begin{cases} \tfrac{1}{2}\epsilon(a_z t - a_t z - a_t v t) + \mu(\tfrac{1}{2}v^2 t + vz) & \text{(Gal)} \\ \tfrac{1}{2}\epsilon[(a_z t - a_t z)\cosh\chi + (a_z z - a_t t)\sinh\chi] & \text{(Poin).} \end{cases} \tag{28}$$

The infinitesimal generators can be calculated from (27) and (28) by differentiation of the operators $U(g)$ in the unit element e of G. The calculation of P_t will be given explicitly as an example:

$$
\begin{aligned}
(P_t\psi)(x) &= -\mathrm{i}\Big(\frac{\partial}{\partial a_t}U(g)\psi\Big)(x)\Big|_{g=e} = -\mathrm{i}\frac{\partial}{\partial a_t}(\exp \mathrm{i}\xi(g, h_{g^{-1}x}))\psi(g^{-1}x)\Big|_{g=e} \\
&= -\mathrm{i}\frac{\partial}{\partial a_t}\psi(g^{-1}x)\Big|_{g=e} + \psi(x)\frac{\partial}{\partial a_t}\xi(g, h_{g^{-1}x})\Big|_{g=e} \\
&= \mathrm{i}\frac{\partial}{\partial t}\psi(x) + \psi(x)\frac{\partial}{\partial a_t}\xi(g, h_x)\Big|_{g=e} = \Big(\mathrm{i}\frac{\partial}{\partial t} + \frac{\partial}{\partial a_t}\xi(g, h_x)\Big|_{g=e}\Big)\psi(x).
\end{aligned}
$$

In this way one obtains the following results for the generators operating on the wavefunctions:

$$
\begin{aligned}
&P_t = \mathrm{i}\partial_t - \tfrac{1}{2}\epsilon z \qquad && P_z = -\mathrm{i}\partial_z - \tfrac{1}{2}\epsilon t \\
&P_x = -\mathrm{i}\partial_x - \tfrac{1}{2}\beta y && P_y = -\mathrm{i}\partial_y + \tfrac{1}{2}\beta x \\
&J = -\mathrm{i}(x\partial_y - y\partial_x) \\
&K = \begin{cases} \mathrm{i}t\partial_z + \mu z \\ \mathrm{i}(t\partial_z + z\partial_t) \end{cases} && \begin{matrix} \text{(Gal)} \\ \text{(Poin).} \end{matrix}
\end{aligned}
\tag{29}
$$

It is easily checked that these expressions obey the commutation relations (9)–(14).

Now we know the form in configuration space of the infinitesimal generators of a projective representation of the invariance group. Substitution of the explicit expressions (29) in equation (17) gives (by an accurate bookkeeping of cross-terms!)

$$
\begin{matrix} \text{(Gal)} \\ \text{(Poin)} \end{matrix} \qquad \left.\begin{matrix} 2\mu(\mathrm{i}\partial_t + \tfrac{1}{2}\epsilon z) \\ (\mathrm{i}\partial_t + \tfrac{1}{2}\epsilon z)^2 \end{matrix}\right\} = (-\mathrm{i}\partial_x + \tfrac{1}{2}\beta y)^2 + (-\mathrm{i}\partial_y - \tfrac{1}{2}\beta x)^2 + (-\mathrm{i}\partial_z + \tfrac{1}{2}\epsilon t)^2 + C_\parallel - C_\perp .
\tag{30}
$$

From these equations it will be clear how the constants ϵ, β, μ, $C_\parallel$ and $C_\perp$ should be interpreted physically. Substitution of

$$
\epsilon = eE, \quad \beta = eB, \quad \mu = m, \quad C_\parallel - C_\perp = \begin{cases} 2m\mathscr{V} & \text{(Gal)} \\ m^2 & \text{(Poin)} \end{cases}
\tag{31}
$$

in equation (30) gives (after division by $2m$ for Galilei) the Schrödinger and Klein–Gordon equations for a particle with mass m and charge e (and internal energy $\mathscr{V}$ for Galilei) in an external uniform field with electric and magnetic vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ along the z axis:

$$
\begin{aligned}
&\mathrm{i}\partial_t - e\Phi = \frac{1}{2m}(-\mathrm{i}\nabla - e\boldsymbol{A})^2 + \mathscr{V} \qquad && \text{(Gal)} \\
&(\mathrm{i}\partial_t - e\Phi)^2 = (-\mathrm{i}\nabla - e\boldsymbol{A})^2 + m^2 && \text{(Poin)}
\end{aligned}
\tag{32}
$$

where

$$
\Phi = -\tfrac{1}{2}Ez, \qquad \boldsymbol{A} = (-\tfrac{1}{2}By, \tfrac{1}{2}Bx, -\tfrac{1}{2}Et).
\tag{33}
$$

The electromagnetic field is present in the equations (32) by minimal coupling to its potential in the so called symmetric gauge (33). This particular gauge is due to our conventions, expecially to our choice of exponents. If we had started from other group exponents, equivalent to (20)–(22), then we would have obtained the equations

(32) with the potential in a different gauge. It can be shown straightforwardly that any gauge can be obtained in this way.

Besides the equations of motion we obtain from (16) the eigenvalue equation

$$(-\mathrm{i}\partial_x - eA_x)^2 + (-\mathrm{i}\partial_y - eA_y)^2 = C_\perp \qquad (34)$$

for the Landau (energy) levels of a charged particle in a magnetic field.

4. Conclusion

In order to convert the group theoretical operator identity (17) into the physical equation of motion (30) one has to know how the infinitesimal generators work in configuration space. In the free particle case this knowledge is provided by a Fourier transformation from momentum to configuration space (Lévy-Leblond 1974, formula 34). In the present case that approach does not work. Here it is equation (23) that contains the information, so the physical foundation of (23) is a crucial point in the derivation. If this foundation were not satisfactory then the sceptical reader, saying that I have put in the minimal coupling by hand through equation (23), would be right. I will show, however, that (23) is satisfactorily founded on quantum mechanical principles.

It is obvious that (23) gives the most general transformation behaviour of a one-component wavefunction $\psi(x)$ under the operators $U(g)$, such that

$$|(U(g)\psi)(gx)|^2 = |\psi(x)|^2 \qquad (35)$$

or, in other words, such that $|\psi(x)|^2$ transforms as a scalar function. So equation (23) is equivalent to (35) which is a direct consequence of the interpretation of $|\psi(x)|^2$ as a probability density. Equation (35) says nothing more than that the probability density $|\psi(x)|^2$ transforms as a scalar function. Hence, it gives expression to the principle of *locality* which in the context of elementary quantum mechanics is a fundamental concept.

By the way, also in the case of a free particle an argument of locality has been implicitly used in the derivation of the equation of motion. The Fourier transformation from momentum to configuration space is a rather *ad hoc* approach. Why not some other unitary transformation? The answer to this question is that one (tacitly) wants to obtain wavefunctions that transform locally, as in (35) and (23). It is only accidental that this result is obtained in the free case by a Fourier transformation. In the present case the expressions (29) for the generators in configuration space cannot be obtained by such a simple transformation (Hoogland 1977, p 88).

The locality principle which is expressed by equation (23) is independent of any specific kind of interaction, and nothing is put in 'by hand' through this equation. Nevertheless, together with the superposition principle and the invariance principle (Lévy-Leblond 1974, 1975) locality implies the minimal electromagnetic coupling for a spinless quantum particle in an external uniform parallel electromagnetic field.

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